Vector analysis 2

Frames 1 to 87

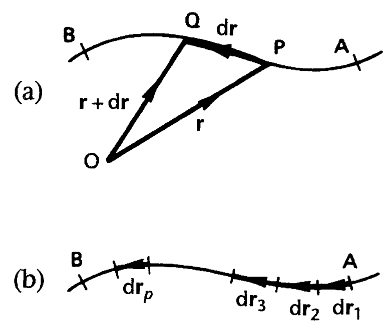
Learning outcomes

When you have completed this Programme you will be able to:

- Evaluate the line integral of a scalar and a vector field in Cartesian coordinates
- Evaluate the volume integral of a vector field
- Evaluate the surface integral of a scalar and a vector field
- Determine whether or not a vector field is a conservative vector field
- Apply Gauss' divergence theorem
- Apply Stokes' theorem
- Determine the direction of unit normal vectors to a surface
- Apply Green's theorem in the plane

We dealt in some detail with line, surface and volume integrals in an earlier Programme, when we approached the subject analytically. In many practical problems, it is more convenient to express these integrals in vector form and the methods often lead to more concise working. 1

Line integrals



Let a point P on the curve c joining A and B be denoted by the position vector $\mathbf{r}$ with respect to a fixed origin O.

If Q is a neighbouring point on the curve with position vector $\mathbf{r} + d\mathbf{r}$, then $\overline{PQ} = d\mathbf{r}$.

The curve c can be divided up into many (n) such small arcs, approximating to $d\mathbf{r}_1$, $d\mathbf{r}_2$, $d\mathbf{r}_3 \ldots d\mathbf{r}_p \ldots$ so that

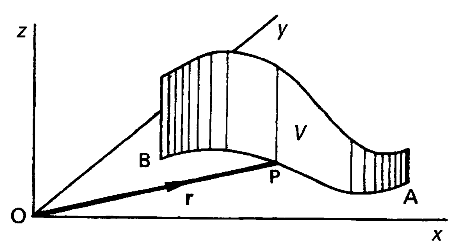
$$\overline{AB} = \sum_{p=1}^{n} d\mathbf{r}_p$$

where $d\mathbf{r}_p$ is a vector representing the element of arc in both magnitude and direction.

Scalar field

If a scalar field V exists for all points on the curve, then $\sum_{p=1}^{n} V\,d\mathbf{r}_p$ with $d\mathbf{r} \to 0$, defines the *line integral* of V along the curve c from A to B,

$$\text{i.e. line integral} = \int_c V\,d\mathbf{r}$$



We can illustrate this integral by erecting a continuous ordinate proportional to V at each point of the curve. $\int_c V\,d\mathbf{r}$ is then represented by the area of the curved surface between the ends A and B of the curve c.

To evaluate a line integral, the integrand is expressed in terms of x, y, z, with $d\mathbf{r} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{dr} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz}$$

In practice, x, y and z are often expressed in terms of parametric equations of a fourth variable (say u), i.e. $x = x(u)$; $y = y(u)$; $z = z(u)$. From these, dx, dy and dz can be written in terms of u and the integral evaluated in terms of this parameter u.

The following examples will show the method.

Example 1

If $V = xy^2z$, evaluate $\int_c V\,\mathbf{dr}$ along the curve c having parametric equations $x = 3u$; $y = 2u^2$; $z = u^3$ between A (0, 0, 0) and B (3, 2, 1).

$$V = xy^2z = (3u)(4u^4)(u^3) = 12u^8$$
$$\mathbf{dr} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz = \ldots\ldots\ldots\ldots$$

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$$\boxed{\mathbf{dr} = \mathbf{i}\,3\,du + \mathbf{j}\,4u\,du + \mathbf{k}\,3u^2\,du}$$

Because

$x = 3u$, $\quad\therefore\ dx = 3\,du$
$y = 2u^2$, $\quad\therefore\ dy = 4u\,du$
$z = u^3$, $\quad\therefore\ dz = 3u^2\,du$

Limits: A (0, 0, 0) corresponds to $u = \ldots\ldots\ldots\ldots$
B (3, 2, 1) corresponds to $u = \ldots\ldots\ldots\ldots$

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$$\boxed{\text{A }(0,0,0) \equiv u = 0 \qquad \text{B }(3,2,1) \equiv u = 1}$$

$$\therefore\ \int_c V\,\mathbf{dr} = \int_0^1 12u^8\,(\mathbf{i}\,3\,du + \mathbf{j}\,4u\,du + \mathbf{k}\,3u^2\,du)$$
$$= \ldots\ldots\ldots\ldots$$

Finish it off

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$$\boxed{4\mathbf{i} + \frac{24}{5}\mathbf{j} + \frac{36}{11}\mathbf{k}}$$

Because

$$\int_c V\,\mathbf{dr} = 12\int_0^1 (\mathbf{i}\,3u^8\,du + \mathbf{j}\,4u^9\,du + \mathbf{k}\,3u^{10}\,du)$$

which integrates directly to give the result quoted above.

Now for another example.

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Example 2

If $V = xy + y^2z$, evaluate $\int_c V\,d\mathbf{r}$ along the curve c defined by $x = t^2$; $y = 2t$; $z = t + 5$ between A (0, 0, 5) and B (4, 4, 7).

As before, expressing V and $d\mathbf{r}$ in terms of the parameter t we have

$V = \ldots\ldots\ldots\ldots \quad d\mathbf{r} = \ldots\ldots\ldots\ldots$

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$$\boxed{V = 6t^3 + 20t^2; \quad d\mathbf{r} = \mathbf{i}\,2t\,dt + \mathbf{j}\,2\,dt + \mathbf{k}\,dt}$$

Because

$$V = xy + y^2z = (t^2)(2t) + (4t^2)(t+5) = 6t^3 + 20t^2.$$

$$\left.\begin{aligned} \text{Also } x &= t^2 & dx &= 2t\,dt \\ y &= 2t & dy &= 2\,dt \\ z &= t+5 & dz &= dt \end{aligned}\right\} \quad \begin{aligned} \therefore\ d\mathbf{r} &= \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz \\ &= \mathbf{i}\,2t\,dt + \mathbf{j}\,2\,dt + \mathbf{k}\,dt \end{aligned}$$

$$\therefore \quad \int_c V\,d\mathbf{r} = \int_c (6t^3 + 20t^2)(\mathbf{i}\,2t + \mathbf{j}\,2 + \mathbf{k})\,dt$$

Limits: A (0, 0, 5) $\equiv t = \ldots\ldots\ldots\ldots$

B (4, 4, 7) $\equiv t = \ldots\ldots\ldots\ldots$

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$$\boxed{\text{A }(0,0,5) \equiv t = 0; \quad \text{B }(4,4,7) \equiv t = 2}$$

$$\therefore \quad \int_c V\,d\mathbf{r} = \int_0^2 (6t^3 + 20t^2)(\mathbf{i}\,2t + \mathbf{j}\,2 + \mathbf{k})\,dt$$

$= \ldots\ldots\ldots\ldots$ Complete the integration.

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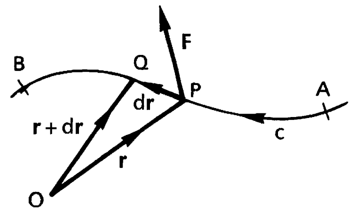
$$\boxed{\frac{8}{15}(444\,\mathbf{i} + 290\,\mathbf{j} + 145\,\mathbf{k})}$$

$$\int_c V\,d\mathbf{r} = 2\int_0^2 \{(6t^4 + 20t^3)\mathbf{i} + (6t^3 + 20t^2)\mathbf{j} + (3t^3 + 10t^2)\mathbf{k}\}\,dt$$

The actual integration is simple enough and gives the result shown. All line integrals in scalar fields are done in the same way.

10 Vector field



If a vector field $\mathbf{F}$ exists for all points of the curve c, then for each element of arc we can form the scalar product $\mathbf{F}\cdot\mathbf{dr}$. Summing these products for all elements of arc, we have $\sum_{p=1}^{n}\mathbf{F}\cdot\mathbf{dr}_p$

Then, if $\mathbf{dr}_p \to 0$, the sum becomes the integral $\int_c \mathbf{F}\cdot\mathbf{dr}$,

i.e. the line integral of $\mathbf{F}$ from A to B along the stated curve

$$= \int_c \mathbf{F}\cdot\mathbf{dr}$$

In this case, since $\mathbf{F}\cdot\mathbf{dr}$ is a scalar product, then the line integral is a scalar.

To evaluate the line integral, $\mathbf{F}$ and $\mathbf{dr}$ are expressed in terms of x, y, z and the curve in parametric form. We have

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

and $$\mathbf{dr} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$$

Then $$\mathbf{F}\cdot\mathbf{dr} = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$
$$= F_x\,dx + F_y\,dy + F_z\,dz$$

$$\therefore \int_c \mathbf{F}\cdot\mathbf{dr} = \int_c F_x\,dx + \int_c F_y\,dy + \int_c F_z\,dz$$

Now for an example to show it in operation.

Example 1

If $\mathbf{F} = x^2y\mathbf{i} + xz\mathbf{j} - 2yz\mathbf{k}$, evaluate $\int_c \mathbf{F}\cdot\mathbf{dr}$ between A (0, 0, 0) and B (4, 2, 1) along the curve having parametric equations $x = 4t$; $y = 2t^2$; $z = t^3$.

Expressing everything in terms of the parameter t, we have

$$\mathbf{F} = \ldots\ldots\ldots\ldots$$

$$dx = \ldots\ldots\ldots\ldots; \quad dy = \ldots\ldots\ldots\ldots; \quad dz = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{array}{c}\mathbf{F} = 32t^4\,\mathbf{i} + 4t^4\,\mathbf{j} - 4t^5\,\mathbf{k} \\ dx = 4\,dt; \quad dy = 4t\,dt; \quad dz = 3t^2\,dt\end{array}}$$

Because

$$\begin{array}{rlll} x^2y &= (16t^2)(2t^2) = 32t^4 & x = 4t & \therefore\ dx = 4\,dt \\ xz &= (4t)(t^3) = 4t^4 & y = 2t^2 & \therefore\ dy = 4t\,dt \\ 2yz &= (4t^2)(t^3) = 4t^5 & z = t^3 & \therefore\ dz = 3t^2\,dt \end{array}$$

Then $\displaystyle\int \mathbf{F}\cdot d\mathbf{r} = \int (32t^4\mathbf{i} + 4t^4\mathbf{j} - 4t^5\mathbf{k})\cdot(\mathbf{i}\,4\,dt + \mathbf{j}\,4t\,dt + \mathbf{k}\,3t^2\,dt)$

$$= \int (128t^4 + 16t^5 - 12t^7)\,dt$$

Limits: A (0, 0, 0) $\equiv t =$ ....; B (4, 2, 1) $\equiv t =$ ....

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$$\boxed{A \equiv t = 0; \quad B \equiv t = 1}$$

$$\therefore \int_c \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (128t^4 + 16t^5 - 12t^7)\,dt = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{128}{5} + \frac{8}{3} - \frac{3}{2} = \frac{803}{30} = 26{\cdot}77}$$

If the vector field $\mathbf{F}$ is a *force field*, then the line integral $\displaystyle\int_c \mathbf{F}\cdot d\mathbf{r}$ represents the work done in moving a unit particle along the prescribed curve c from A to B.

Now for another example.

Example 2

If $\mathbf{F} = x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}$, evaluate $\displaystyle\int_c \mathbf{F}\cdot d\mathbf{r}$ between A (0, 0, 0) and B (1, 2, 3)

(a) along the straight lines c_1 from (0, 0, 0) to (1, 0, 0)
then c_2 from (1, 0, 0) to (1, 2, 0)
and c_3 from (1, 2, 0) to (1, 2, 3)

(b) along the straight line c_4 joining (0, 0, 0) to (1, 2, 3).

As before, we first obtain an expression for $\mathbf{F}\cdot d\mathbf{r}$ which is

....

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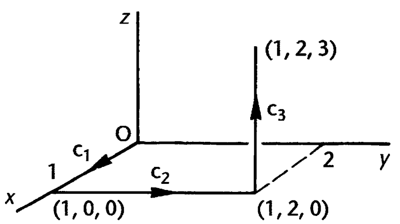
$$\boxed{\mathbf{F}\cdot d\mathbf{r} = x^2y\,dx + 2yz\,dy + 3z^2x\,dz}$$

Because

$$\mathbf{F}\cdot d\mathbf{r} = (x^2y\,\mathbf{i} + 2yz\,\mathbf{j} + 3z^2x\,\mathbf{k})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

$$\therefore \quad \int \mathbf{F}\cdot d\mathbf{r} = \int x^2y\,dx + \int 2yz\,dy + \int 3z^2x\,dz$$

(a) Here the integration is made in three sections, along c_1, c_2 and c_3.



(1) c_1: $y = 0,\ z = 0,\ dy = 0,\ dz = 0$

$$\therefore \quad \int_{c_1} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

(2) c_2: The conditions along c_2 are

....

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$$\boxed{c_2:\quad x = 1,\quad z = 0,\quad dx = 0,\quad dz = 0}$$

$$\therefore \quad \int_{c_2} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

(3) c_3: $x = 1,\ y = 2,\ dx = 0,\ dy = 0$

$$\therefore \quad \int_{c_3} \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$$\boxed{27}$$

Because

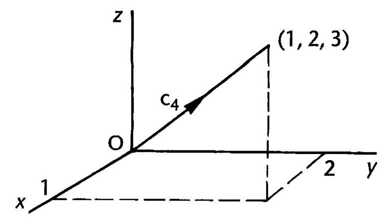
$$\int_{c_3} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + \int_0^3 3z^2\,dz = 27$$

Summing the three partial results

$$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + 27 = 27 \quad \therefore \quad \int_{c_1+c_2+c_3} \mathbf{F}\cdot d\mathbf{r} = 27$$

(b) If t is taken as the parameter, the parametric equations of c are

$x = \ldots\ldots\ldots\ldots$
$y = \ldots\ldots\ldots\ldots$
$z = \ldots\ldots\ldots\ldots$



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$$\boxed{x = t; \quad y = 2t: \quad z = 3t}$$

and the limits of t are $\ldots\ldots\ldots\ldots$

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$$\boxed{t = 0 \text{ and } t = 1}$$

As in Example 1, we now express everything in terms of t and complete the integral, finally getting

$$\int_{c_4} \mathbf{F} \cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_{c_4} \mathbf{F} \cdot d\mathbf{r} = \frac{115}{4} = 28{\cdot}75}$$

Because

$$\mathbf{F} = 2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k}$$
$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz = \mathbf{i}\,dt + \mathbf{j}\,2\,dt + \mathbf{k}\,3\,dt$$

$$\therefore \int_{c_4} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\,dt$$
$$= \int_0^1 (2t^3 + 24t^2 + 81t^3)\,dt = \int_0^1 (83t^3 + 24t^2)\,dt$$
$$= \left[83\frac{t^4}{4} + 8t^3\right]_0^1 = \frac{115}{4} = 28{\cdot}75$$

So the value of the line integral depends on the path taken between the two end points A and B

(a) $\int \mathbf{F} \cdot d\mathbf{r}$ via c_1, c_2 and $c_3 = 27$

(b) $\int \mathbf{F} \cdot d\mathbf{r}$ via c_4 $= 28{\cdot}75$

We shall refer to this topic later.

One further example on your own. The working is just the same as before.

▶

Example 3

If $\mathbf{F} = x^2y^2\mathbf{i} + y^3z\mathbf{j} + z^2\mathbf{k}$, evaluate $\int_c \mathbf{F}\cdot d\mathbf{r}$ along the curve $x = 2u^2$, $y = 3u$, $z = u^3$ between A $(2, -3, -1)$ and B $(2, 3, 1)$. Proceed as before. You will have no difficulty.

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \dots\dots\dots\dots$$

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$$\boxed{\int_c \mathbf{F}\cdot d\mathbf{r} = \frac{500}{21} = 23{\cdot}8}$$

Here is the working for you to check.

$$x = 2u^2 \quad y = 3u \quad z = u^3$$

$$\begin{aligned} x^2y^2 &= (4u^4)(9u^2) = 36u^6 & dx &= 4u\,du \\ y^3z &= (27u^3)(u^3) = 27u^6 & dy &= 3\,du \\ z^2 &= u^6 & dz &= 3u^2\,du \end{aligned}$$

Limits: A $(2, -3, -1)$ corresponds to $u = -1$
B $(2, 3, 1)$ corresponds to $u = 1$

$$\therefore \int_c \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} (x^2y^2\mathbf{i} + y^3z\mathbf{j} + z^2\mathbf{k})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

$$= \int_{-1}^{1} (36u^6\mathbf{i} + 27u^6\mathbf{j} + u^6\mathbf{k})\cdot(\mathbf{i}\,4u\,du + \mathbf{j}\,3\,du + \mathbf{k}\,3u^2\,du)$$

$$= \int_{-1}^{1} (144u^7 + 81u^6 + 3u^8)\,du$$

$$= \left[18u^8 + \frac{81u^7}{7} + \frac{u^9}{3}\right]_{-1}^{1} = \frac{500}{21} = 23{\cdot}8$$

Now on to the next section

Volume integrals

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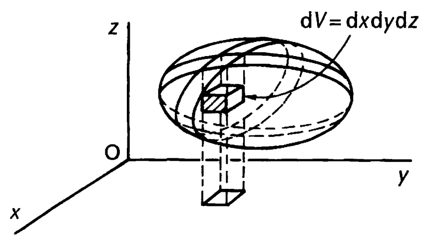
If V is a closed region bounded by a surface $\mathbf{S}$ and $\mathbf{F}$ is a vector field at each point of V and on its boundary surface $\mathbf{S}$, then $\int_V \mathbf{F}\,dV$ is the *volume integral* of $\mathbf{F}$ throughout the region.



$$\int_V \mathbf{F}\,dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \mathbf{F}\,dz\,dy\,dx$$

▶

Example 1

Evaluate $\int_V \mathbf{F}\,\mathrm{d}V$ where V is the region bounded by the planes $x = 0$, $x = 2$, $y = 0$, $y = 3$, $z = 0$, $z = 4$, and $\mathbf{F} = xy\mathbf{i} + z\mathbf{j} - x^2\mathbf{k}$.

We start, as in most cases, by sketching the diagram, which is

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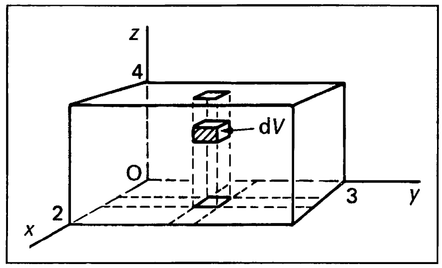

Then $\mathbf{F} = xy\,\mathbf{i} + z\,\mathbf{j} - x^2\,\mathbf{k}$ and $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

$$\therefore \quad \int_V \mathbf{F}\,\mathrm{d}V = \int_0^4 \int_0^3 \int_0^2 (xy\mathbf{i} + z\mathbf{j} - x^2\mathbf{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^4 \int_0^3 \left[\frac{x^2y}{2}\mathbf{i} + xz\mathbf{j} - \frac{x^3}{3}\mathbf{k}\right]_{x=0}^{x=2} \mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^4 \int_0^3 \left(2y\mathbf{i} + 2z\mathbf{j} - \frac{8}{3}\mathbf{k}\right) \mathrm{d}y\,\mathrm{d}z$$

$= \ldots\ldots\ldots\ldots$ Complete the integral.

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$$\boxed{\int_V \mathbf{F}\,\mathrm{d}V = 4(9\mathbf{i} + 12\mathbf{j} - 8\mathbf{k})}$$

Because

$$\int_V \mathbf{F}\,\mathrm{d}V = \int_0^4 \left[y^2\mathbf{i} + 2yz\mathbf{j} - \frac{8}{3}y\mathbf{k}\right]_{y=0}^{y=3} \mathrm{d}z$$

$$= \int_0^4 (9\mathbf{i} + 6z\mathbf{j} - 8\mathbf{k})\,\mathrm{d}z$$

$$= \left[9z\mathbf{i} + 3z^2\mathbf{j} - 8z\mathbf{k}\right]_0^4$$

$$= 36\mathbf{i} + 48\mathbf{j} - 32\mathbf{k}$$

$$= 4(9\mathbf{i} + 12\mathbf{j} - 8\mathbf{k})$$

Now another.

▶

Example 2

Evaluate $\int_V \mathbf{F}\,dV$ where V is the region bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+y+z=2$, and $\mathbf{F}=2z\mathbf{i}+y\mathbf{k}$.

To sketch the surface $2x+y+z=2$, note that

when $z=0$, $\quad 2x+y=2 \quad$ i.e. $y=2-2x$
when $y=0$, $\quad 2x+z=2 \quad$ i.e. $z=2-2x$
when $x=0$, $\quad y+z=2 \quad$ i.e. $z=2-y$

Inserting these in the planes $x=0$, $y=0$, $z=0$ will help.
The diagram is therefore

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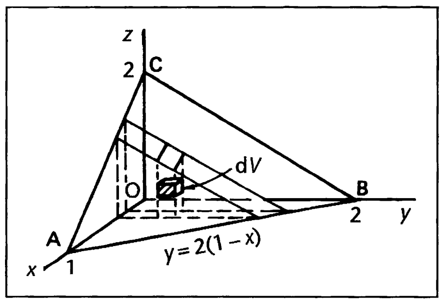

So $2x+y+z=2$ cuts the axes at A (1, 0, 0); B (0, 2, 0); C (0, 0, 2).

Also $\mathbf{F}=2z\mathbf{i}+y\mathbf{k}$; $\quad z=2-2x-y=2(1-x)-y$

$$\therefore \int_V \mathbf{F}\,dV=\int_0^1\int_0^{2(1-x)}\int_0^{2(1-x)-y}(2z\mathbf{i}+y\mathbf{k})\,dz\,dy\,dx$$

$$=\int_0^1\int_0^{2(1-x)}\Big[z^2\mathbf{i}+yz\mathbf{k}\Big]_{z=0}^{z=2(1-x)-y}dy\,dx$$

$$=\int_0^1\int_0^{2(1-x)}\{[4(1-x)^2-4(1-x)y+y^2]\mathbf{i}$$
$$+[2(1-x)y-y^2]\mathbf{k}\}\,dy\,dx$$

$$=\int_0^1\Bigg[\left\{4(1-x)^2y-2(1-x)y^2+\frac{y^3}{3}\right\}\mathbf{i}$$
$$+\left\{(1-x)y^2-\frac{y^3}{3}\right\}\mathbf{k}\Bigg]_{y=0}^{2(1-x)}dx$$

$= . . . .$

Finish the last stage

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$$\boxed{\int_V \mathbf{F}\,\mathrm{d}V = \frac{1}{3}(2\mathbf{i}+\mathbf{k})}$$

Because

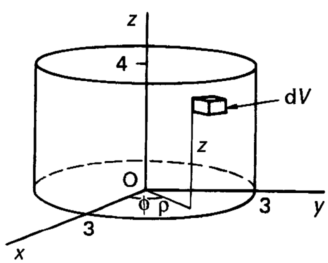
$$\int_V \mathbf{F}\,\mathrm{d}V = \int_0^1 \left\{\frac{8}{3}(1-x)^3\mathbf{i} + \frac{4}{3}(1-x)^3\mathbf{k}\right\}\mathrm{d}x$$

$$= \left[-\frac{2}{3}(1-x)^4\mathbf{i} - \frac{1}{3}(1-x)^4\mathbf{k}\right]_0^1 = \frac{1}{3}(2\mathbf{i}+\mathbf{k})$$

And now one more, slightly different.

Example 3

Evaluate $\int_V \mathbf{F}\,\mathrm{d}V$ where $\mathbf{F} = 2\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$ and V is the region bounded by the planes $z = 0$, $z = 4$ and the surface $x^2 + y^2 = 9$.



It will be convenient to use cylindrical polar coordinates (ρ, ϕ, z) so the relevant transformations are

$x = \ldots\ldots\ldots\ldots$; $\quad y = \ldots\ldots\ldots\ldots$

$z = \ldots\ldots\ldots\ldots$; $\quad \mathrm{d}V = \ldots\ldots\ldots\ldots$

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$x = \rho\cos\phi$;	$y = \rho\sin\phi$
$z = z$;	$\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$

Then $\int_V \mathbf{F}\,\mathrm{d}V = \iiint_V (2\mathbf{i} + 2z\mathbf{j} + y\mathbf{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Changing into cylindrical polar coordinates with appropriate change of limits this becomes

$$\int_V \mathbf{F}\,\mathrm{d}V = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{3}\int_{z=0}^{4}(2\mathbf{i} + 2z\mathbf{j} + \rho\sin\phi\,\mathbf{k})\,\mathrm{d}z\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{3}\left[2z\mathbf{i} + z^2\mathbf{j} + \rho\sin\phi\,z\mathbf{k}\right]_{z=0}^{4}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$= \int_0^{2\pi}\int_0^3 (8\mathbf{i} + 16\mathbf{j} + 4\rho\sin\phi\,\mathbf{k})\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$= 4\int_0^{2\pi}\int_0^3 (2\rho\mathbf{i} + 4\rho\mathbf{j} + \rho^2\sin\phi\,\mathbf{k})\,\mathrm{d}\rho\,\mathrm{d}\phi$$

Completing the working, we finally get

$$\int_V \mathbf{F}\,\mathrm{d}V = \ldots\ldots\ldots\ldots$$

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$$\boxed{72\pi(\mathbf{i}+2\mathbf{j})}$$

Because

$$\int_V \mathbf{F}\,dV = 4\int_0^{2\pi}\left[\rho^2\mathbf{i}+2\rho^2\mathbf{j}+\frac{\rho^3}{3}\sin\phi\,\mathbf{k}\right]_0^3 d\phi$$
$$= 4\int_0^{2\pi}(9\mathbf{i}+18\mathbf{j}+9\sin\phi\,\mathbf{k})\,d\phi$$
$$= 36\int_0^{2\pi}(\mathbf{i}+2\mathbf{j}+\sin\phi\,\mathbf{k})\,d\phi$$
$$= 36\Big[\phi\mathbf{i}+2\phi\mathbf{j}-\cos\phi\,\mathbf{k}\Big]_0^{2\pi}$$
$$= 36\{(2\pi\mathbf{i}+4\pi\mathbf{j}-\mathbf{k})-(-\mathbf{k})\}$$
$$= 72\pi(\mathbf{i}+2\mathbf{j})$$

You will, of course, remember that in appropriate cases, the use of cylindrical polar coordinates or spherical polar coordinates often simplifies the subsequent calculations. So keep them in mind.

Now let us turn to surface integrals – in the next frame

Surface integrals

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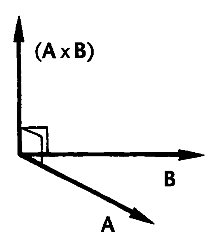
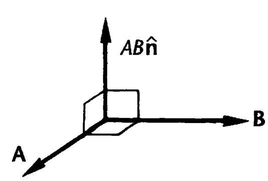
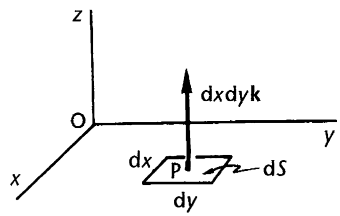


The vector product of two vectors **A** and **B** has magnitude $|\mathbf{A}\times\mathbf{B}| = AB\sin\theta$ at right angles to the plane of **A** and **B** to form a right-handed set.



If $\theta = \frac{\pi}{2}$, then $|\mathbf{A}\times\mathbf{B}| = AB$ in the direction of the normal. Therefore, if $\hat{\mathbf{n}}$ is a unit normal then

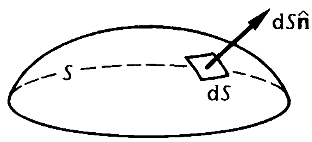
$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\hat{\mathbf{n}} = AB\,\hat{\mathbf{n}}$$



If P (x, y) is a point in the x–y plane, the element of area $dx\,dy$ has a vector area $\mathbf{dS} = (\mathbf{i}\,dx) \times (\mathbf{j}\,dy)$.

i.e. $\mathbf{dS} = dx\,dy(\mathbf{i} \times \mathbf{j}) = dx\,dy\,\mathbf{k}$

i.e. a vector of magnitude $dx\,dy$ acting in the direction of $\mathbf{k}$ and referred to as the *vector area*.



For a general surface S in space, each element of surface dS has a *vector area* dS such that $\mathbf{dS} = dS\,\hat{\mathbf{n}}$.

You will remember we established previously that for a surface S given by the equation $\phi(x, y, z) = \text{constant}$, the unit normal $\hat{\mathbf{n}}$ is given by

$$\hat{\mathbf{n}} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{\nabla\phi}{|\nabla\phi|}$$

Let us see how we can apply these results to the following examples.

Scalar fields

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Example 1

A scalar field $V = xyz$ exists over the curved surface S defined by $x^2 + y^2 = 4$ between the planes $z = 0$ and $z = 3$ in the first octant. Evaluate $\int_S V\,\mathbf{dS}$ over this surface.

We have $V = xyz \quad S\colon\ x^2 + y^2 - 4 = 0, \ z = 0 \ \text{to} \ z = 3$

$$\mathbf{dS} = \hat{\mathbf{n}}\,dS \qquad \text{where } \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

Now $\nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j}$ and

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{4} = 4$$

Therefore

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{x\mathbf{i} + y\mathbf{j}}{2} \quad \text{so that } \mathbf{dS} = \hat{\mathbf{n}}\,dS = \frac{x\mathbf{i} + y\mathbf{j}}{2}\,dS$$

$$\therefore \int_S V\,\mathbf{dS} = \int_S V\,\hat{\mathbf{n}}\,dS$$

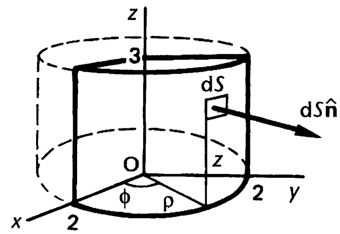
$$= \frac{1}{2}\int_S xyz(x\mathbf{i} + y\mathbf{j})\,dS$$

$$= \frac{1}{2}\int_S (x^2yz\mathbf{i} + xy^2z\mathbf{j})\,dS \qquad (1)$$

▶

We have to evaluate this integral over the prescribed surface.

Changing to cylindrical coordinates with $\rho = 2$



$x = \dots\dots\dots\dots;\qquad y = \dots\dots\dots\dots$
$z = \dots\dots\dots\dots;\qquad \mathrm{d}S = \dots\dots\dots\dots$

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$$\boxed{\begin{array}{ll} x = 2\cos\phi; & y = 2\sin\phi \\ z = z; & \mathrm{d}S = 2\,\mathrm{d}\phi\,\mathrm{d}z \end{array}}$$

$$\begin{aligned} \therefore\ x^2yz &= (4\cos^2\phi)(2\sin\phi)(z) \\ &= 8\cos^2\phi\sin\phi\, z \\ xy^2z &= (2\cos\phi)(4\sin^2\phi)(z) \\ &= 8\cos\phi\sin^2\phi\, z \end{aligned}$$

Then result (1) above becomes

$$\begin{aligned} \int_S V\,\mathrm{d}\mathbf{S} &= \frac{1}{2}\int_0^{\pi/2}\int_0^3 (8\cos^2\phi\sin\phi\, z\mathbf{i} + 8\cos\phi\sin^2\phi\, z\mathbf{j})\,2\,\mathrm{d}z\,\mathrm{d}\phi \\ &= 4\int_0^{\pi/2}\int_0^3 (\cos^2\phi\sin\phi\,\mathbf{i} + \cos\phi\sin^2\phi\,\mathbf{j})\,2z\,\mathrm{d}z\,\mathrm{d}\phi \\ &= 4\int_0^{\pi/2} (\cos^2\phi\sin\phi\,\mathbf{i} + \cos\phi\sin^2\phi\,\mathbf{j})\,9\,\mathrm{d}\phi \end{aligned}$$

and this eventually gives

$$\int_S V\,\mathrm{d}\mathbf{S} = \dots\dots\dots\dots$$

$$\boxed{\int_S V\,\mathrm{d}\mathbf{S} = 12(\mathbf{i}+\mathbf{j})}$$

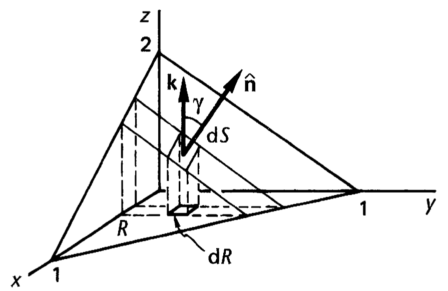
Because

$$\int_S V\,\mathrm{d}\mathbf{S} = 36\left[-\frac{\cos^3\phi}{3}\mathbf{i} + \frac{\sin^3\phi}{3}\mathbf{j}\right]_0^{\pi/2} = 12(\mathbf{i}+\mathbf{j})$$

Example 2

A scalar field $V = x + y + z$ exists over the surface S defined by $2x + 2y + z = 2$ bounded by $x = 0,\ y = 0,\ z = 0$ in the first octant.

Evaluate $\int_S V\,\mathrm{d}\mathbf{S}$ over this surface.



S: $2x + 2y + z = 2$

$x = 0 \qquad z = 2 - 2y$

$y = 0 \qquad z = 2 - 2x$

$z = 0 \qquad y = 1 - x$

$$\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S \qquad \text{where } \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

Now $\nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$|\nabla\phi| = \sqrt{4+4+1} = \sqrt{9} = 3$$

Therefore

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{3} \quad \text{so that } \mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S = \frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})\,\mathrm{d}S$$

If we now project $\mathrm{d}S$ onto the x–y plane, $\mathrm{d}R = \mathrm{d}S\cos\gamma$

$$\cos\gamma = \hat{\mathbf{n}}\cdot\mathbf{k} = \frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(\mathbf{k}) = \frac{1}{3}$$

$$\therefore\ \mathrm{d}R = \frac{1}{3}\mathrm{d}S \qquad \therefore\ \mathrm{d}S = 3\mathrm{d}R = 3\,\mathrm{d}x\mathrm{d}y$$

$$\therefore\ \int_S V\,\mathrm{d}\mathbf{S} = \int_S V\hat{\mathbf{n}}\,\mathrm{d}S = \int_S\!\!\int (x+y+z)\frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})3\,\mathrm{d}x\,\mathrm{d}y$$

But $z = 2 - 2x - 2y$

$$\therefore\ \int_S V\,\mathrm{d}\mathbf{S} = \int_{x=0}^{1}\int_{y=0}^{1-x}(2-x-y)(2\mathbf{i}+2\mathbf{j}+\mathbf{k})\,\mathrm{d}y\,\mathrm{d}x$$

$$= \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{2}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})}$$

Because

$$\int_S V\,\mathrm{d}\mathbf{S} = \int_0^1 \left[2y - xy - \frac{y^2}{2}\right]_0^{1-x} (2\mathbf{i}+2\mathbf{j}+\mathbf{k})\,\mathrm{d}x$$
$$= \left[\frac{3}{2}x - x^2 + \frac{x^3}{6}\right]_0^1 (2\mathbf{i}+2\mathbf{j}+\mathbf{k})$$
$$= \frac{2}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})$$

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Vector fields

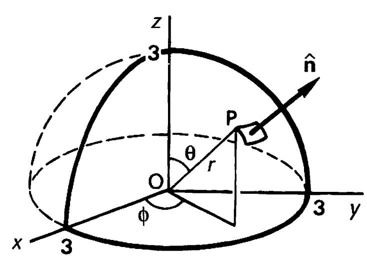
Example 1

A vector field $\mathbf{F} = y\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ exists over a surface S defined by $x^2 + y^2 + z^2 = 9$ bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant. Evaluate $\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ over the surface indicated.

$$\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S \qquad \text{where } \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} \text{ where } \phi = x^2 + y^2 + z^2 - 9 = 0$$

Now $\nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{9} = 6$$



$$\therefore\ \hat{\mathbf{n}} = \frac{1}{6}(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$$
$$= \frac{1}{3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_S (y\mathbf{i} + 2\mathbf{j} + \mathbf{k})\cdot\frac{1}{3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,\mathrm{d}S$$
$$= \frac{1}{3}\int_S (xy + 2y + z)\,\mathrm{d}S$$

Before integrating over the surface, we convert to spherical polar coordinates.

$x = \ldots\ldots\ldots\ldots;\qquad y = \ldots\ldots\ldots\ldots$

$z = \ldots\ldots\ldots\ldots;\qquad \mathrm{d}S = \ldots\ldots\ldots\ldots$

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$$x = 3\sin\theta\cos\phi; \qquad y = 3\sin\theta\sin\phi$$
$$z = 3\cos\theta; \qquad dS = 9\sin\theta\, d\theta\, d\phi$$

Limits of θ and ϕ are $\theta = 0$ to $\frac{\pi}{2}$; $\phi = 0$ to $\frac{\pi}{2}$.

$$\therefore \int_S \mathbf{F}\cdot d\mathbf{S} = \frac{1}{3}\int_0^{\pi/2}\int_0^{\pi/2}(9\sin^2\theta\sin\phi\cos\phi + 6\sin\theta\sin\phi + 3\cos\theta)\, 9\sin\theta\, d\theta\, d\phi$$
$$= 9\int_0^{\pi/2}\int_0^{\pi/2}(3\sin^3\theta\sin\phi\cos\phi + 2\sin^2\theta\sin\phi + \sin\theta\cos\theta)\, d\theta\, d\phi$$
$$= \dots\dots\dots\dots$$

Complete the integral

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$$\int_S \mathbf{F}\cdot d\mathbf{S} = 9\left(1 + \frac{3\pi}{4}\right)$$

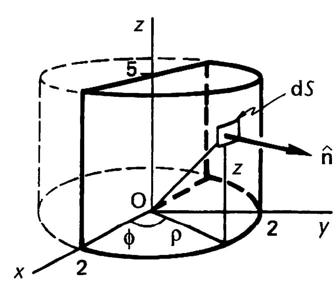
Because

$$\int_S \mathbf{F}\cdot d\mathbf{S} = 9\int_0^{\pi/2}\left(2\sin\phi\cos\phi + \frac{\pi}{2}\sin\phi + \frac{1}{2}\right) d\phi$$
$$= 9\left[\sin^2\phi - \frac{\pi}{2}\cos\phi - \frac{\phi}{2}\right]_0^{\pi/2} = 9\left(1 + \frac{3\pi}{4}\right)$$

Example 2

Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$ and S is the surface $x^2 + y^2 = 4$ in the first two octants bounded by the planes $z = 0$, $z = 5$ and $y = 0$.

ϕ: $x^2 + y^2 - 4 = 0 \qquad \hat{\mathbf{n}} = \dfrac{\nabla\phi}{|\nabla\phi|}$



$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j}$$
$$\therefore |\nabla\phi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2}$$
$$= 2\sqrt{4} = 4$$
$$\therefore \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{4} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j})$$
$$\therefore \int_S \mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$$

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$$\boxed{\int_S y^2\,dS}$$

Because

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_S (2y\mathbf{j}+z\mathbf{k})\cdot\frac{1}{2}(x\mathbf{i}+y\mathbf{j})\,dS$$
$$= \frac{1}{2}\int_S (2y^2)\,dS = \int_S y^2\,dS$$

This is clearly a case for using cylindrical polar coordinates.

$x = \dots\dots\dots\dots;\qquad y = \dots\dots\dots\dots$
$z = \dots\dots\dots\dots;\qquad dS = \dots\dots\dots\dots$

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$$\boxed{\begin{array}{ll} x = 2\cos\phi; & y = 2\sin\phi \\ z = z; & dS = 2\,d\phi\,dz \end{array}}$$

$$\therefore\quad \int_S \mathbf{F}\cdot d\mathbf{S} = \int_S y^2\,dS = \int_S\int 4\sin^2\phi\, 2\,d\phi\,dz = 8\int_S\int \sin^2\phi\,d\phi\,dz$$

Limits: $\phi = 0$ to $\phi = \pi$; $z = 0$ to $z = 5$

$$\therefore\quad \int_S \mathbf{F}\cdot d\mathbf{S} = \dots\dots\dots\dots$$

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$$\boxed{20\pi}$$

Because

$$\int_S \mathbf{F}\cdot dS = 4\int_{z=0}^{5}\int_{\phi=0}^{\pi}(1-\cos 2\phi)\,d\phi\,dz$$
$$= 4\int_0^5\left[\phi - \frac{\sin 2\phi}{2}\right]_0^{\pi} dz$$
$$= 4\int_0^5 \pi\,dz = 4\pi\Big[z\Big]_0^5 = 20\pi$$

Example 3

Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}$ is the field $x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}$ and S is the surface $2x + y + 2z = 2$ bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant.

We can sketch the diagram by putting $x = 0$, $y = 0$, $z = 0$ in turn in the equation for S.

When	$x = 0$	$y + 2z = 2$	$z = 1 - \frac{y}{2}$
	$y = 0$	$x + z = 1$	$z = 1 - x$
	$z = 0$	$2x + y = 2$	$y = 2 - 2x$

So the diagram is

$\dots\dots\dots\dots$

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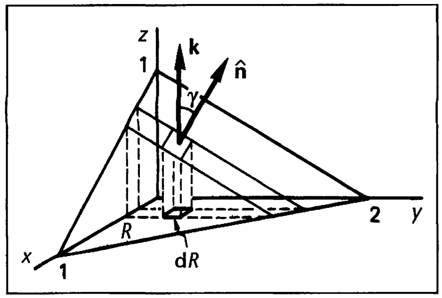

$$\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}; \qquad \phi: \quad 2x + y + 2z - 2 = 0$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \qquad |\nabla\phi| = 3$$

$$\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

$$= \ldots\ldots\ldots\ldots \text{ (next stage)}$$

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$$\boxed{\frac{1}{3}\int_S (2x^2 - y + 4z)\,\mathrm{d}S}$$

Because

$$\int_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \int_S (x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}) \cdot \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})\,\mathrm{d}S$$

$$= \frac{1}{3}\int_S (2x^2 - y + 4z)\,\mathrm{d}S$$

If we now project the element of surface dS onto the x–y plane

$$\mathrm{d}R = \mathrm{d}S\cos\gamma \quad \cos\gamma = \hat{\mathbf{n}} \cdot \mathbf{k} \quad \therefore\ \mathrm{d}R = \hat{\mathbf{n}} \cdot \mathbf{k}\,\mathrm{d}S \quad \therefore\ \mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{\hat{\mathbf{n}} \cdot \mathbf{k}}$$

$$\therefore\ \hat{\mathbf{n}} \cdot \mathbf{k} = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{k}) = \frac{2}{3} \qquad \therefore\ \mathrm{d}S = \frac{3}{2}\,\mathrm{d}x\,\mathrm{d}y$$

Using these new relationships, $\displaystyle\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$

$$= \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_R \int \frac{1}{2}(2x^2 - y + 4z)\,dx\,dy}$$

Because

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{3}\int_S (2x^2 - y + 4z)\,dS$$
$$= \frac{1}{3}\int_R \int (2x^2 - y + 4z)\frac{3}{2}\,dx\,dy$$
$$= \frac{1}{2}\int_R \int (2x^2 - y + 4z)\,dx\,dy$$

Limits: $y = 0$ to $y = 2 - 2x$; $x = 0$ to $x = 1$

$$\therefore \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{2}\int_0^1 \int_0^{2-2x} (2x^2 - y + 4z)\,dy\,dx$$

But $2x + y + 2z = 2 \quad \therefore\ z = \dfrac{1}{2}(2 - 2x - y)$

$$\therefore \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \ldots\ldots\ldots\ldots$$

Complete the integration

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$$\boxed{\frac{1}{2}}$$

Here is the rest of the working.

$$\int_S \mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{2}\int_0^1 \int_0^{2-2x} (2x^2 - y + 4 - 4x - 2y)\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 \int_0^{2-2x} (2x^2 - 4x + 4 - 3y)\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 \left[(2x^2 - 4x + 4)y - \frac{3y^2}{2}\right]_0^{2-2x} dx$$
$$= \frac{1}{2}\int_0^1 (4x^2 - 8x + 8 - 4x^3 + 8x^2 - 8x - 6 + 12x - 6x^2)\,dx$$
$$= \frac{1}{2}\int_0^1 (6x^2 - 4x^3 - 4x + 2)\,dx = \int_0^1 (3x^2 - 2x^3 - 2x + 1)\,dx$$
$$= \left[x^3 - \frac{x^4}{2} - x^2 + x\right]_0^1 = \frac{1}{2}$$

While we are concerned with vector fields, let us move on to a further point of interest.

Conservative vector fields

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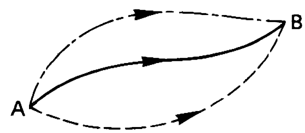
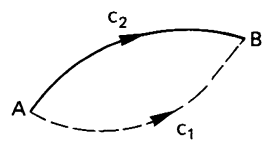
In general, the value of the line integral $\int_c \mathbf{F}\cdot d\mathbf{r}$ between two stated points A and B depends on the particular path of integration followed.



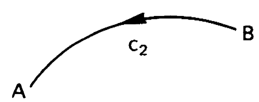
If, however, the line integral between A and B is independent of the path of integration between the two end points, then the vector field $\mathbf{F}$ is said to be *conservative*.

It follows that, for a closed path in a conservative field, $\oint_c \mathbf{F}\cdot d\mathbf{r} = 0$.

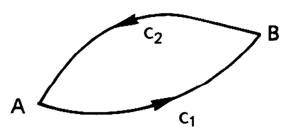


Because, if the field is conservative

$$\int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} = \int_{c_2(AB)} \mathbf{F}\cdot d\mathbf{r}$$



$$\text{But } \int_{c_2(BA)} \mathbf{F}\cdot d\mathbf{r} = -\int_{c_2(AB)} \mathbf{F}\cdot d\mathbf{r}$$



Hence, for the closed path $\mathbf{AB}_{c_1} + \mathbf{BA}_{c_2}$

$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} + \int_{c_2(BA)} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} - \int_{c_2(AB)} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} - \int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} = 0$$

$$\therefore \quad \oint \mathbf{F}\cdot d\mathbf{r} = 0$$

Note that this result holds good only for a closed curve and when the vector field is a conservative field.

Now for an example.

Example

If $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$, evaluate the line integral $\int \mathbf{F}\cdot d\mathbf{r}$ between A (0, 0, 0) and B (2, 4, 6)

(a) along the curve c whose parametric equations are $x = u$, $y = u^2$, $z = 3u$

(b) along the three straight lines c_1: (0, 0, 0) to (2, 0, 0); c_2: (2, 0, 0) to (2, 4, 0); c_3: (2, 4, 0) to (2, 4, 6).

Hence determine whether or not $\mathbf{F}$ is a conservative field.

First draw the diagram

. . . .

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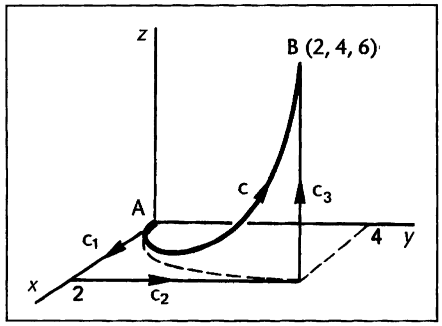

(a) $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$

$$x = u; \qquad y = u^2; \qquad z = 3u$$

$$\therefore \; dx = du; \qquad dy = 2u\,du; \qquad dz = 3\,du.$$

$$\mathbf{F}\cdot d\mathbf{r} = (2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

$$= 2xyz\,dx + x^2z\,dy + x^2y\,dz$$

Using the transformations shown above, we can now express $\mathbf{F}\cdot d\mathbf{r}$ in terms of u.

$$\mathbf{F}\cdot d\mathbf{r} = \dots\dots\dots\dots$$

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$$\boxed{15u^4\,du}$$

Because

$$2xyz\,dx = (2u)(u^2)(3u)\,du = 6u^4\,du$$

$$x^2z\,dy = (u^2)(3u)(2u)\,du = 6u^4\,du$$

$$x^2y\,dz = (u^2)(u^2)3\,du = 3u^4\,du$$

$$\therefore \; \mathbf{F}\cdot d\mathbf{r} = 6u^4\,du + 6u^4\,du + 3u^4\,du = 15u^4\,du$$

The limits of integration in u are

$$\dots\dots\dots\dots$$

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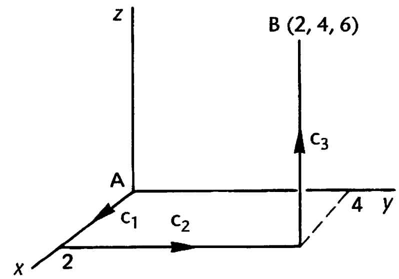
$$\boxed{u = 0 \text{ to } u = 2}$$

$$\therefore \int_c \mathbf{F} \cdot d\mathbf{r} = \int_0^2 15u^4 \, du = \left[3u^5\right]_0^2 = 96 \qquad \int_c \mathbf{F} \cdot d\mathbf{r} = 96$$

(b) The diagram for (b) is as shown. We consider each straight line section in turn.



$$\int \mathbf{F} \cdot d\mathbf{r} = \int (2xyz \, dx + x^2 z \, dy + x^2 y \, dz)$$

c_1: $(0, 0, 0)$ to $(2, 0, 0)$; $y = 0$, $z = 0$, $dy = 0$, $dz = 0$

$$\therefore \int_{c_1} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

In the same way, we evaluate the line integral along c_2 and c_3.

$$\int_{c_2} \mathbf{F} \cdot d\mathbf{r} = \ldots\ldots\ldots\ldots; \qquad \int_{c_3} \mathbf{F} \cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_{c_2} \mathbf{F} \cdot d\mathbf{r} = 0; \qquad \int_{c_3} \mathbf{F} \cdot d\mathbf{r} = 96}$$

Because we have $\int \mathbf{F} \cdot d\mathbf{r} = \int (2xyz \, dx + x^2 z \, dy + x^2 y \, dz)$

c_2: $(2, 0, 0)$ to $(2, 4, 0)$; $x = 2$, $z = 0$, $dx = 0$, $dz = 0$

$$\therefore \int_{c_2} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

$$\int_{c_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

c_3: $(2, 4, 0)$ to $(2, 4, 6)$; $x = 2$, $y = 4$, $dx = 0$, $dy = 0$

$$\therefore \int_{c_3} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + \int_0^6 16 \, dz = \left[16z\right]_0^6 = 96$$

$$\int_{c_3} \mathbf{F} \cdot d\mathbf{r} = 96$$

Collecting the three results together

$$\int_{c_1 + c_2 + c_3} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 96 \qquad \therefore \int_{c_1 + c_2 + c_3} \mathbf{F} \cdot d\mathbf{r} = 96$$

▶

In this particular example, the value of the line integral is independent of the two paths we have used joining the same two end points and indicates that **F** may be a conservative field. It follows that

$$\int_C \mathbf{F}\cdot d\mathbf{r} - \int_{C_1+C_2+C_3} \mathbf{F}\cdot d\mathbf{r} = 0 \quad \text{i.e.} \quad \oint \mathbf{F}\cdot d\mathbf{r} = 0$$

So, if **F** is a conservative field, $\oint \mathbf{F}\cdot d\mathbf{r} = 0$

Make a note of this for future use

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Two tests can be applied to establish that a given vector field is conservative.

If **F** is a conservative field

(a) curl **F** = 0

(b) **F** can be expressed as grad V where V is a scalar field to be determined.

For example, in the work we have just completed, we showed that $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ is a conservative field.

(a) If we determine curl **F** in this case, we have

curl **F** = ...

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$$\boxed{\text{curl } \mathbf{F} = 0}$$

Because

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix}$$

$$= (x^2 - x^2)\mathbf{i} - (2xy - 2xy)\mathbf{j} + (2xz - 2xz)\mathbf{k} = \mathbf{0}$$

$\therefore$ curl $\mathbf{F} = \mathbf{0}$

(b) We can attempt to express **F** as grad V where V is a scalar in x, y, z.

If $V = f(x, y, z)$

$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}$$

and we have $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$

$$\therefore \quad \frac{\partial V}{\partial x} = 2xyz \qquad \therefore \quad V = x^2yz + f(y, z)$$

$$\frac{\partial V}{\partial y} = x^2z \qquad \therefore \quad V = \ldots\ldots\ldots\ldots$$

$$\frac{\partial V}{\partial z} = x^2y \qquad \therefore \quad V = \ldots\ldots\ldots\ldots$$

We therefore have to find a scalar function V that satisfies the three requirements. $V = \ldots\ldots\ldots\ldots$

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$$\boxed{V = x^2yz}$$

Because

$$\frac{\partial V}{\partial x} = 2xyz \qquad \therefore\; V = x^2yz + f(y, z)$$

$$\frac{\partial V}{\partial y} = x^2z \qquad \therefore\; V = x^2yz + g(x, z)$$

$$\frac{\partial V}{\partial z} = x^2y \qquad \therefore\; V = x^2yz + h(x, y)$$

These three are satisfied if $f(y, z) = g(z, x) = h(x, y) = 0$

$\therefore\; \mathbf{F} = \text{grad } V$ where $V = x^2yz$

So two tests can be applied to determine whether or not a vector field is conservative. They are

(a) ....
(b) ....

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(a) curl $\mathbf{F} = 0$
(b) $\mathbf{F} =$ grad V

Any one of these conditions can be applied as is convenient.

Now what about these?

Exercise

Determine which of the following vector fields are conservative.

(a) $\mathbf{F} = (x + y)\mathbf{i} + (y - z)\mathbf{j} + (x + y + z)\mathbf{k}$
(b) $\mathbf{F} = (2xz + y)\mathbf{i} + (z + x)\mathbf{j} + (x^2 + y)\mathbf{k}$
(c) $\mathbf{F} = y\sin z\,\mathbf{i} + x\sin z\,\mathbf{j} + (xy\cos z + 2z)\mathbf{k}$
(d) $\mathbf{F} = 2xy\mathbf{i} + (x^2 + 4yz)\mathbf{j} + 2y^2z\mathbf{k}$
(e) $\mathbf{F} = y\cos x\cos z\,\mathbf{i} + \sin x\cos z\,\mathbf{j} - y\sin x\sin z\,\mathbf{k}$.

Complete all five and check your findings with the next frame.

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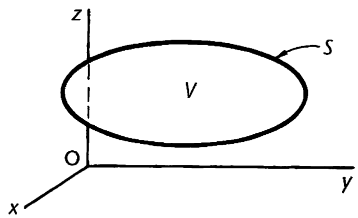
(a) No (b) Yes (c) Yes (d) No (e) Yes

Divergence theorem (Gauss' theorem)



For a closed surface S, enclosing a region V in a vector field $\mathbf{F}$,

$$\int_V \operatorname{div} \mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

In general, this means that the volume integral (triple integral) on the left-hand side can be expressed as a surface integral (double integral) on the right-hand side. Let us work through one or two examples.

Example 1

Verify the divergence theorem for the vector field $\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ taken over the region bounded by the planes $z = 0$, $z = 2$, $x = 0$, $x = 1$, $y = 0$, $y = 3$.

Start off, as always, by sketching the relevant diagram, which is

....

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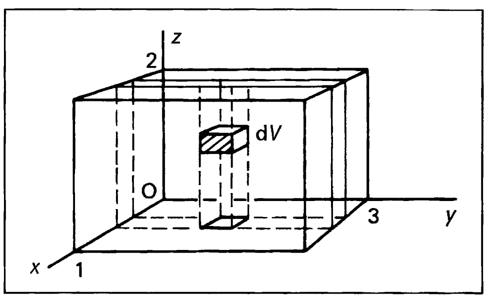

$dV = dx\,dy\,dz$

We have to show that

$$\int_V \operatorname{div} \mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

(a) To find $\int_V \operatorname{div} \mathbf{F}\,dV$

$$\operatorname{div} \mathbf{F} = \nabla\cdot\mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\,\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(y) = 2x + 0 + 0 = 2x$$

$$\therefore \int_V \operatorname{div} \mathbf{F}\,dV = \int_V 2x\,dV = \iiint_V 2x\,dz\,dy\,dx$$

Inserting the limits and completing the integration

$$\int_V \operatorname{div} \mathbf{F}\,dV = \text{...........}$$

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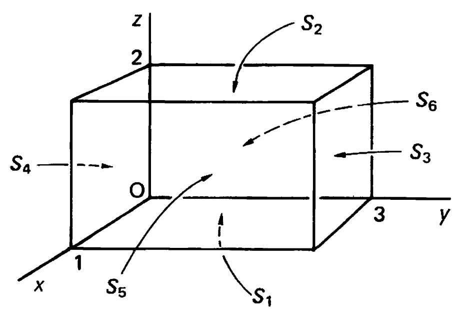
$$\boxed{\int_V \operatorname{div} \mathbf{F}\, dV = 6}$$

Because

$$\int_V \operatorname{div} \mathbf{F}\, dV = \int_0^1 \int_0^3 \int_0^2 2x\, dz\, dy\, dx = \int_0^1 \int_0^3 \Big[2xz\Big]_0^2 dy\, dx$$
$$= \int_0^1 \Big[4xy\Big]_0^3 dx = \int_0^1 12x\, dx = \Big[6x^2\Big]_0^1 = 6$$

Now we have to find $\int_S \mathbf{F} \cdot d\mathbf{S}$

(b) To find $\int_S \mathbf{F} \cdot d\mathbf{S}$ i.e. $\int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$



The enclosing surface S consists of six separate plane faces denoted as $S_1, S_2, \dots, S_6$ as shown. We consider each face in turn.

$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$

(1) S_1 (base): $z = 0$; $\hat{\mathbf{n}} = -\mathbf{k}$ (outwards and downwards)

$$\therefore \; \mathbf{F} = x^2\mathbf{i} + y\mathbf{k} \qquad dS_1 = dx\, dy$$

$$\therefore \int_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int\int_{S_1} (x^2\mathbf{i} + y\mathbf{k}) \cdot (-\mathbf{k})\, dy\, dx$$
$$= \int_0^1 \int_0^3 (-y)\, dy\, dx$$
$$= \int_0^1 \left[-\frac{y^2}{2}\right]_0^3 dx$$
$$= -\frac{9}{2}$$

(2) S_2 (top): $z = 2$; $\quad \hat{\mathbf{n}} = \mathbf{k} \qquad dS_2 = dx\, dy$

$$\therefore \int_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$$

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$$\boxed{\frac{9}{2}}$$

Because

$$\int_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int\int_{S_2} (x^2\mathbf{i} + 2\mathbf{j} + y\mathbf{k}) \cdot (\mathbf{k})\, dy\, dx$$

$$= \int_0^1 \int_0^3 y\, dy\, dx = \frac{9}{2}$$

So we go on.

(3) S_3 (right-hand end): $y = 3$; $\quad \hat{\mathbf{n}} = \mathbf{j} \quad dS_3 = dx\, dz$

$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$

$$\therefore \int_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int\int_{S_3} (x^2\mathbf{i} + z\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{j})\, dz\, dx$$

$$= \int_0^1 \int_0^2 z\, dz\, dx$$

$$= \int_0^1 \left[\frac{z^2}{2}\right]_0^2 dx = \int_0^1 2\, dx = 2$$

(4) S_4 (left-hand end): $y = 0$; $\quad \hat{\mathbf{n}} = -\mathbf{j} \quad dS_4 = dx\, dz$

$$\therefore \int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$$

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$$\boxed{-2}$$

Because

$$\int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int\int_{S_4} (x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}) \cdot (-\mathbf{j})\, dz\, dx = \int_0^1 \int_0^2 (-z)\, dz\, dx$$

$$= \int_0^1 \left[-\frac{z^2}{2}\right]_0^2 dx = \int_0^1 (-2)\, dx = -2$$

Now for the remaining two sides S_5 and S_6.
Evaluate these in the same manner, obtaining

$$\int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$$

$$\int_{S_6} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$$

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$$\boxed{\int_{S_5} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 6; \qquad \int_{S_6} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 0}$$

Check:

(5) S_5 (front): $x = 1$; $\hat{\mathbf{n}} = \mathbf{i}$ $dS_5 = dy\,dz$

$$\therefore \int_{S_5} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_5} (\mathbf{i} + z\mathbf{j} + y\mathbf{k})\cdot(\mathbf{i})\,dy\,dz = \iint_{S_5} 1\,dy\,dz = 6$$

(6) S_6 (back): $x = 0$; $\hat{\mathbf{n}} = -\mathbf{i}$ $dS_6 = dy\,dz$

$$\therefore \int_{S_6} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_6} (z\mathbf{j} + y\mathbf{k})\cdot(-\mathbf{i})\,dy\,dz = \iint_{S_6} 0\,dy\,dz = 0$$

Now on to the next frame where we will collect our results together

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For the whole surface S we therefore have

$$\int_S \mathbf{F}\cdot d\mathbf{S} = -\frac{9}{2} + \frac{9}{2} + 2 - 2 + 6 + 0 = 6$$

and from our previous work in section (a) $\int_V \operatorname{div}\mathbf{F}\,dV = 6$

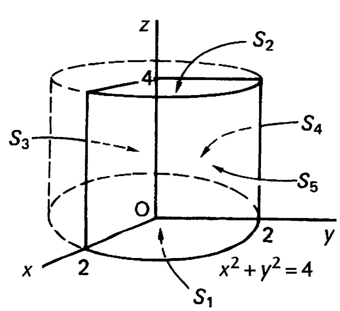
We have therefore verified as required that, in this example

$$\int_V \operatorname{div}\mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

We have made rather a meal of this since we have set out the working in detail. In practice, the actual writing can often be considerably simplified. Let us move on to another example.

Example 2

Verify the Gauss divergence theorem for the vector field $\mathbf{F} = x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by the planes $z = 0$, $z = 4$, $x = 0$, $y = 0$ and the surface $x^2 + y^2 = 4$ in the first octant.



Divergence theorem

$$\int_V \operatorname{div}\mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

S consists of five surfaces $S_1, S_2, \ldots, S_5$ as shown.

(a) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k})$

$= \ldots\ldots\ldots\ldots$

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$$\boxed{1+2z}$$

$$\therefore \int_V \operatorname{div} \mathbf{F}\, dV = \int_V \nabla \cdot \mathbf{F}\, dV = \iiint_V (1+2z)\, dx\, dy\, dz$$

Changing to cylindrical polar coordinates (ρ, ϕ, z)

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \qquad dV = \rho\, d\rho\, d\phi\, dz$$

Transforming the variables and inserting the appropriate limits, we then have

$$\int_V \operatorname{div} \mathbf{F}\, dV = \ldots\ldots\ldots\ldots$$

Finish it

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$$\boxed{20\pi}$$

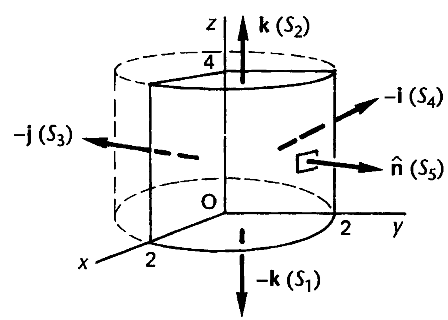
Because

$$\int_V \operatorname{div} \mathbf{F}\, dV = \int_0^{\pi/2}\int_0^2\int_0^4 (1+2z)\, dz\, \rho\, d\rho\, d\phi$$

$$= \int_0^{\pi/2}\int_0^2 \left[z+z^2\right]_0^4 \rho\, d\rho\, d\phi = \int_0^{\pi/2}\int_0^2 20\rho\, d\rho\, d\phi$$

$$= \int_0^{\pi/2}\left[10\rho^2\right]_0^2 d\phi = \int_0^{\pi/2} 40\, d\phi = 20\pi \qquad (1)$$

(b) Now we evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ over the closed surface.



The unit normal vector for each surface is shown.

$$\mathbf{F} = x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}$$

(1) S_1: $z = 0$; $\quad \hat{\mathbf{n}} = -\mathbf{k} \quad \mathbf{F} = x\mathbf{i} + 2\mathbf{j}$

$$\therefore \int_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \int_{S_1} (x\mathbf{i}+2\mathbf{j})\cdot(-\mathbf{k})\, dS = 0$$

(2) S_2: $z = 4$; $\quad \hat{\mathbf{n}} = \mathbf{k} \quad \mathbf{F} = x\mathbf{i} + 2\mathbf{j} + 16\mathbf{k}$

$$\therefore \int_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \int_{S_2} (x\mathbf{i} + 2\mathbf{j} + 16\mathbf{k}) \cdot (\mathbf{k}) \, dS = \int_{S_2} 16 \, dS$$
$$= 16\left(\frac{\pi 4}{4}\right) = 16\pi$$

In the same way for S_3: $\displaystyle\int_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \dots\dots\dots\dots$

and for S_4: $\displaystyle\int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \dots\dots\dots\dots$

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$$\boxed{\int_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = -16; \qquad \int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = 0}$$

Because we have

(3) S_3: $y = 0$; $\quad \hat{\mathbf{n}} = -\mathbf{j} \quad \mathbf{F} = x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}$

$$\therefore \int_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \int_{S_3} (x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}) \cdot (-\mathbf{j}) \, dS$$
$$= \int_{S_3} (-2) \, dS = -2(8) = -16$$

(4) S_4: $x = 0$; $\quad \hat{\mathbf{n}} = -\mathbf{i} \quad \mathbf{F} = 2\mathbf{j} + z^2\mathbf{k}$

$$\therefore \int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \int_{S_4} (2\mathbf{j} + z^2\mathbf{k}) \cdot (-\mathbf{i}) \, dS = 0$$

Finally we have

(5) S_5: $x^2 + y^2 - 4 = 0 \quad \hat{\mathbf{n}} = \dots\dots\dots\dots$

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$$\boxed{\hat{\mathbf{n}} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j})}$$

Because

$$x^2 + y^2 - 4 = 0 \qquad \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\mathbf{i} + y\mathbf{j}}{2}$$

$$\therefore \int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \int_{S_5} (x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}) \cdot \left(\frac{x\mathbf{i} + y\mathbf{j}}{2}\right) dS = \frac{1}{2}\int_{S_5} (x^2 + 2y) \, dS$$

Converting to cylindrical polar coordinates, this gives

$$\int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \dots\dots\dots\dots$$

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$$\boxed{4\pi + 16}$$

Because we have

$$\int_{S_5} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{2}\int_{S_5} (x^2 + 2y)\,dS$$

also $\quad x = 2\cos\phi; \qquad y = 2\sin\phi$

$\quad z = z; \qquad dS = 2\,d\phi\,dz$

$$\therefore \int_{S_5} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{2}\int_0^4\int_0^{\pi/2} (4\cos^2\phi + 4\sin\phi)\,2\,d\phi\,dz$$

$$= 2\int_0^4\int_0^{\pi/2} \{(1 + \cos 2\phi) + 2\sin\phi\}\,d\phi\,dz$$

$$= 2\int_0^4 \left[\left(\phi - \frac{\sin 2\phi}{2}\right) - 2\cos\phi\right]_0^{\pi/2} dz$$

$$= 2\int_0^4 \left(\frac{\pi}{2} + 2\right) dz = 4\pi + 16$$

Therefore, for the total surface S

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 0 + 16\pi - 16 + 0 + 4\pi + 16 = 20\pi \tag{2}$$

$$\therefore \int_V \operatorname{div}\mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S} = 20\pi$$

Other examples are worked in much the same way. You will remember that, for a closed surface, the normal vectors at all points are drawn in an *outward* direction.

Now we move on to a further important theorem.

Stokes' theorem

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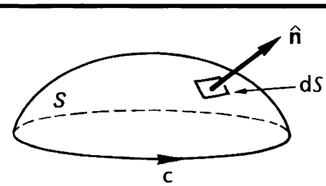


If $\mathbf{F}$ is a vector field existing over an open surface S and around its boundary, closed curve c, then

$$\int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

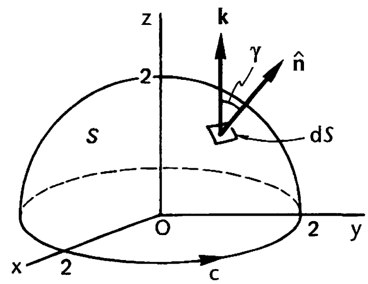
This means that we can express a surface integral in terms of a line integral round the boundary curve.

The proof of this theorem is rather lengthy and is to be found in the Appendix. Let us demonstrate its application in the following examples.

Example 1

A hemisphere S is defined by $x^2 + y^2 + z^2 = 4$ $(z \geq 0)$. A vector field $\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$ exists over the surface and around its boundary c.

Verify Stokes' theorem, that $\int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$.



S: $x^2 + y^2 + z^2 - 4 = 0$
$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$
c is the circle $x^2 + y^2 = 4$.

(a) $$\oint_c \mathbf{F} \cdot d\mathbf{r} = \int_c (2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}) \cdot (\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$
$$= \int_c (2y\,dx - x\,dy + xz\,dz)$$

Converting to polar coordinates

$x = 2\cos\theta$; $\quad y = 2\sin\theta$; $\quad z = 0$
$dx = -2\sin\theta\,d\theta$; $\quad dy = 2\cos\theta\,d\theta$; $\quad$ Limits $\theta = 0$ to 2π

Making the substitutions and completing the integral

$$\oint_c \mathbf{F} \cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\oint_c \mathbf{F} \cdot d\mathbf{r} = -12\pi}$$

Because

$$\oint_c \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (4\sin\theta[-2\sin\theta\,d\theta] - 2\cos\theta\,2\cos\theta\,d\theta)$$
$$= -4\int_0^{2\pi} (2\sin^2\theta + \cos^2\theta)\,d\theta$$
$$= -4\int_0^{2\pi} (1 + \sin^2\theta)\,d\theta = -2\int_0^{2\pi} (3 - \cos 2\theta)\,d\theta$$
$$= -2\left[3\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = -12\pi \qquad (1)$$

On to the next frame

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(b) Now we determine $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$

$$\int \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS \qquad \mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$$

$$\therefore\ \text{curl}\,\mathbf{F} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\text{curl}\,\mathbf{F} = -z\mathbf{j} - 3\mathbf{k}}$$

Because

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y & -x & xz \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}(z-0) + \mathbf{k}(-1-2) = -z\mathbf{j} - 3\mathbf{k}$$

$$\text{Now } \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{2}$$

$$\text{Then } \int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_S (-z\mathbf{j} - 3\mathbf{k})\cdot\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{2}\right)dS$$

$$= \frac{1}{2}\int_S (-yz - 3z)\,dS$$

Expressing this in spherical polar coordinates and integrating, we get

$$\int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \ldots\ldots\ldots\ldots$$

68

$$\boxed{-12\pi}$$

Because

$x = 2\sin\theta\,\cos\phi;\quad y = 2\sin\theta\sin\phi;\quad z = 2\cos\theta;\quad dS = 4\sin\theta\,d\theta\,d\phi$

$$\therefore\ \int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{2}\int_S\int(-2\sin\theta\sin\phi\,2\cos\theta - 6\cos\theta)4\,\sin\theta\,d\theta\,d\phi$$

$$= -4\int_0^{2\pi}\int_0^{\pi/2}(2\sin^2\theta\cos\theta\sin\phi + 3\sin\theta\,\cos\theta)\,d\theta\,d\phi$$

$$= -4\int_0^{2\pi}\left[\frac{2\sin^3\theta\sin\phi}{3} + \frac{3\sin^2\theta}{2}\right]_0^{\pi/2}d\phi$$

$$= -4\int_0^{2\pi}\left(\frac{2}{3}\sin\phi + \frac{3}{2}\right)d\phi = -12\pi \qquad (2)$$

So we have from our two results (1) and (2)

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Before we proceed with another example, let us clarify a point relating to the direction of unit normal vectors now that we are dealing with surfaces.

So on to the next frame

Direction of unit normal vectors to a surface *S*

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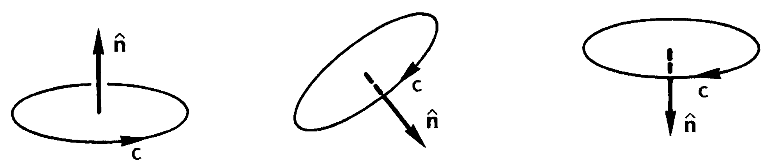
When we were dealing with the divergence theorem, the normal vectors were drawn in a direction outward from the enclosed region.

With an open surface as we now have, there is in fact no inward or outward direction. With any general surface, a normal vector can be drawn in either of two opposite directions. To avoid confusion, a convention must therefore be agreed upon and the established rule is as follows.

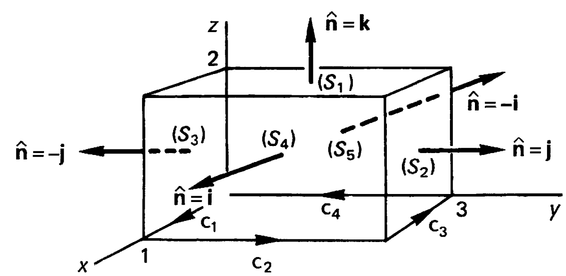


A unit normal $\hat{\mathbf{n}}$ is drawn perpendicular to the surface S at any point in the direction indicated by applying a right-handed screw sense to the direction of integration round the boundary c.

Having noted that point, we can now deal with the next example.

Example 2

A surface consists of five sections formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 3$, $z = 2$ in the first octant. If the vector field $\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} + xy\mathbf{k}$ exists over the surface and around its boundary, verify Stokes' theorem.



If we progress round the boundary along c_1, c_2, c_3, c_4 in an anticlockwise manner, the normals to the surfaces will be as shown.

We have to verify that $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$

(a) We will start off by finding $\oint_c \mathbf{F}\cdot d\mathbf{r}$

$$\int \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int \mathbf{F}\cdot d\mathbf{r} = \int (y\,dx + z^2\,dy + xy\,dz)}$$

(1) Along c_1: $y = 0;\quad z = 0;\quad dy = 0;\quad dz = 0$

$$\therefore \int_{c_1} \mathbf{F}\cdot d\mathbf{r} = \int (0+0+0) = 0$$

(2) Along c_2: $x = 1;\quad z = 0;\quad dx = 0;\quad dz = 0$

$$\therefore \int_{c_2} \mathbf{F}\cdot d\mathbf{r} = \int (0+0+0) = 0$$

In the same way

$$\int_{c_3} \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots \quad \text{and} \quad \int_{c_4} \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_{c_3} \mathbf{F}\cdot d\mathbf{r} = -3; \quad \int_{c_4} \mathbf{F}\cdot d\mathbf{r} = 0}$$

Because

(3) Along c_3: $y = 3;\quad z = 0;\quad dy = 0;\quad dz = 0$

$$\therefore \int_{c_3} \mathbf{F}\cdot d\mathbf{r} = \int_1^0 (3\,dx + 0 + 0) = \Big[3x\Big]_1^0 = -3$$

(4) Along c_4: $x = 0;\quad z = 0;\quad dx = 0;\quad dz = 0$

$$\therefore \int_{c_4} \mathbf{F}\cdot d\mathbf{r} = \int (0+0+0) = 0$$

$$\therefore \oint_c \mathbf{F}\cdot d\mathbf{r} = 0 + 0 - 3 + 0 = -3$$

$$\oint_c \mathbf{F}\cdot d\mathbf{r} = -3 \qquad (1)$$

(b) Now we have to find $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$.

First we need an expression for curl $\mathbf{F}$.

$$\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} + xy\mathbf{k}$$

$$\therefore \text{ curl}\,\mathbf{F} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\text{curl } \mathbf{F} = (x - 2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}}$$

Because

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z^2 & xy \end{vmatrix}$$

$$= \mathbf{i}(x - 2z) - \mathbf{j}(y - 0) + \mathbf{k}(0 - 1) = (x - 2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}$$

Then, for each section, we obtain $\int \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$

(1) S_1 (top): $\hat{\mathbf{n}} = \mathbf{k}$

$$\therefore \int_{S_1} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \ldots\ldots\ldots\ldots$$

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$$\boxed{-3}$$

Because

$$\int_{S_1} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_1} \{(x - 2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\} \cdot (\mathbf{k})\, dS$$

$$= \int_{S_1} (-1)\, dS = -(\text{area of } S_1) = -3$$

Then, likewise

(2) S_2 (right-hand end): $\hat{\mathbf{n}} = \mathbf{j}$

$$\therefore \int_{S_2} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_2} \{(x - 2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\} \cdot (\mathbf{j})\, dS$$

$$= \int_{S_2} (-y)\, dS$$

But $y = 3$ for this section

$$\therefore \int_{S_2} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_2} (-3)\, dS = (-3)(2) = -6$$

(3) S_3 (left-hand end): $\hat{\mathbf{n}} = -\mathbf{j}$

$$\therefore \int_{S_3} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, d\mathbf{S} = \ldots\ldots\ldots\ldots$$

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$$\boxed{0}$$

Because

$$\int_{S_3} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = \int_{S_3} \{(x-2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\}\cdot(-\mathbf{j})\,\text{d}S$$

$$= \int_{S_3} y\,\text{d}S$$

But $y = 0$ over S_3

$$\therefore \int_{S_3} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = 0$$

Working in the same way

$$\int_{S_4} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = \ldots\ldots\ldots\ldots; \quad \int_{S_5} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_{S_4} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = -6; \quad \int_{S_5} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = 12}$$

Because

(4) S_4 (front): $\hat{\mathbf{n}} = \mathbf{i}$

$$\therefore \int_{S_4} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = \int_{S_4} \{(x-2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\}\cdot(\mathbf{i})\,\text{d}S$$

$$= \int_{S_4} (x-2z)\,\text{d}S$$

But $x = 1$ over S_4

$$\therefore \int_{S_4} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = \int_0^3\int_0^2 (1-2z)\,\text{d}z\,\text{d}y = \int_0^3 \Big[z - z^2\Big]_0^2\,\text{d}y$$

$$= \int_0^3 (-2)\,\text{d}y = \Big[-2y\Big]_0^3 = -6$$

(5) S_5 (back): $\hat{\mathbf{n}} = -\mathbf{i}$ with $x = 0$ over S_5

Similar working to that above gives $\int_{S_5} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = 12$

Finally, collecting the five results together gives

$$\int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\text{d}S = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -3-6+0-6+12=-3} \qquad (2)$$

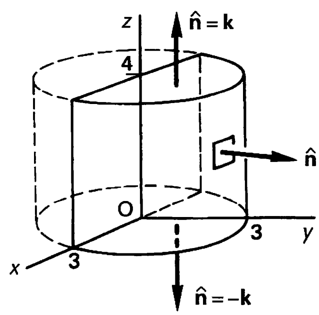
So, referring back to our result for section (a) we see that

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Of course we can, on occasions, make use of Stokes' theorem to lighten the working – as in the next example.

Example 3

A surface S consists of that part of the cylinder $x^2+y^2=9$ between $z=0$ and $z=4$ for $y\geq 0$ and the two semicircles of radius 3 in the planes $z=0$ and $z=4$. If $\mathbf{F}=z\mathbf{i}+xy\mathbf{j}+xz\mathbf{k}$, evaluate $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ over the surface.



The surface S consists of three sections

(a) the curved surface of the cylinder
(b) the top and bottom semicircles.

We could therefore evaluate

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$$

over each of these separately.

However, we know by Stokes' theorem that

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \ldots\ldots\ldots\ldots$$

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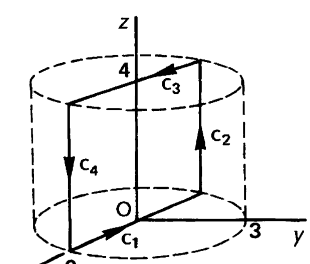
$$\boxed{\oint_c \mathbf{F}\cdot d\mathbf{r} \text{ where c is the boundary of } S}$$



$$\mathbf{F}=z\mathbf{i}+xy\mathbf{j}+xz\mathbf{k}$$

$$\therefore \oint_c \mathbf{F}\cdot d\mathbf{r} = \oint_c (z\mathbf{i}+xy\mathbf{j}+xz\mathbf{k})\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz)$$

$$= \oint_c (z\,dx+xy\,dy+xz\,dz)$$

Now we can work through this easily enough, taking c_1, c_2, c_3, c_4 in turn, and summing the results, which gives

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$\boxed{-24}$

Here is the working in detail. $\oint_c \mathbf{F}\cdot d\mathbf{r} = \oint_c (z\,dx + xy\,dy + xz\,dz)$

(1) c_1: $y = 0$; $z = 0$; $dy = 0$; $dz = 0$

$$\int_{c_1} \mathbf{F}\cdot d\mathbf{r} = \int_{c_1} (0+0+0) = 0$$

(2) c_2: $x = -3$; $y = 0$; $dx = 0$; $dy = 0$

$$\int_{c_2} \mathbf{F}\cdot d\mathbf{r} = \int_{c_2} (0+0-3z\,dz) = \left[\frac{-3z^2}{2}\right]_0^4 = -24$$

(3) c_3: $y = 0$; $z = 4$; $dy = 0$; $dz = 0$

$$\int_{c_3} \mathbf{F}\cdot d\mathbf{r} = \int_{c_3} (4\,dx+0+0) = \int_{-3}^{3} 4\,dx = 24$$

(4) c_4: $x = 3$; $y = 0$; $dx = 0$; $dy = 0$

$$\int_{c_4} \mathbf{F}\cdot d\mathbf{r} = \int_{c_4} (0+0+3z\,dz) = \left[\frac{3z^2}{2}\right]_4^0 = -24$$

Totalling up these four results, we have

$$\oint_c \mathbf{F}\cdot d\mathbf{r} = 0 - 24 + 24 - 24 = -24$$

But $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r} \quad \therefore \int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = -24$

This working is a good deal easier than calculating $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ over the three separate surfaces direct.

So, if you have not already done so, make a note of Stokes' theorem:

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Then on to the next section of the work

Green's theorem

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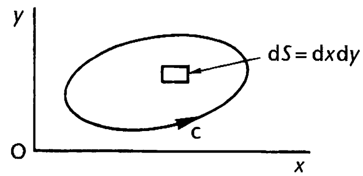
Green's theorem enables an integral over a plane area to be expressed in terms of a line integral round its boundary curve.

We showed in Programme 14 that, if P and Q are two single-valued functions of x and y, continuous over a plane surface S, and c is its boundary curve, then

$$\oint_{c}(P\,dx + Q\,dy) = \int\int_{S}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

where the line integral is taken round c in an anticlockwise manner.

In vector terms, this becomes:



S is a two-dimensional space enclosed by a simple closed curve c.

$$dS = dx\,dy$$
$$d\mathbf{S} = \hat{\mathbf{n}}\,dS = \mathbf{k}\,dx\,dy$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ where $P = P(x, y)$ and $Q = Q(x, y)$ then

curl $\mathbf{F} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}$$

Because

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$= \mathbf{i}\left(0 - \frac{\partial Q}{\partial z}\right) - \mathbf{j}\left(0 - \frac{\partial P}{\partial z}\right) + \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

But in the x–y plane, $\dfrac{\partial Q}{\partial z} = \dfrac{\partial P}{\partial z} = 0$. $\therefore$ curl $\mathbf{F} = \mathbf{k}\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)$

So $\int \text{curl } \mathbf{F}\cdot d\mathbf{S} = \int \text{curl } \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ and in the x–y plane, $\hat{\mathbf{n}} = \mathbf{k}$

$$\therefore \int_{S}\text{curl } \mathbf{F}\cdot d\mathbf{S} = \int_{S}\mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot(\mathbf{k})\,dS = \int\int_{S}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$\therefore \int_{S}\text{curl } \mathbf{F}\cdot d\mathbf{S} = \int\int_{S}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy \qquad (1)$$

Now by Stokes' theorem $\ldots\ldots\ldots\ldots$

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$$\boxed{\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}}$$

and, in this case, $\oint_c \mathbf{F}\cdot d\mathbf{r} = \oint_c (P\mathbf{i} + Q\mathbf{j})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$

$$= \oint_c (P\,dx + Q\,dy)$$

$$\therefore \quad \oint_c \mathbf{F}\cdot d\mathbf{r} = \oint_c (P\,dx + Q\,dy) \qquad (2)$$

Therefore from (1) and (2)

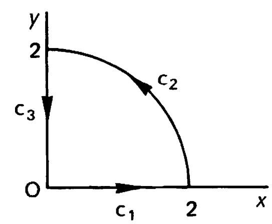
Stokes' theorem $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$ in two dimensions becomes

Green's theorem $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_c (P\,dx + Q\,dy)$

Example

Verify Green's theorem for the integral $\oint_c \{(x^2 + y^2)\,dx + (x + 2y)\,dy\}$ taken round the boundary curve c defined by

$$\begin{aligned} y &= 0 & 0 &\le x \le 2 \\ x^2 + y^2 &= 4 & 0 &\le x \le 2 \\ x &= 0 & 0 &\le y \le 2. \end{aligned}$$



Green's theorem: $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_c (P\,dx + Q\,dy)$

In this case $(x^2 + y^2)\,dx + (x + 2y)\,dy = P\,dx + Q\,dy$

$\therefore\ P = x^2 + y^2$ and $Q = x + 2y$

We now take c_1, c_2, c_3 in turn.

(1) c_1: $y = 0$; $dy = 0$

$$\therefore \quad \int_{c_1} (P\,dx + Q\,dy) = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

(2) c_2: $x^2 + y^2 = 4 \quad \therefore\ y^2 = 4 - x^2 \quad \therefore\ y = (4 - x^2)^{1/2}$

$$x + 2y = x + 2(4 - x^2)^{1/2}$$

$$dy = \frac{1}{2}(4 - x^2)^{-1/2}(-2x)\,dx = \frac{-x}{\sqrt{4 - x^2}}\,dx$$

$$\therefore \quad \int_{c_2} (P\,dx + Q\,dy) = \ldots\ldots\ldots\ldots$$

Make any necessary substitutions and evaluate the line integral for c_2.

$$\boxed{\pi - 4}$$

Because we have

$$\int_{c_2}(P\,dx + Q\,dy) = \int_{c_2}\left\{4 + (x + 2\sqrt{4-x^2})\left(\frac{-x}{\sqrt{4-x^2}}\right)\right\}dx$$

$$= \int_{c_2}\left\{4 - 2x - \frac{x^2}{\sqrt{4-x^2}}\right\}dx$$

Putting $x = 2\sin\theta$, $\sqrt{4-x^2} = 2\cos\theta$ $\quad dx = 2\cos\theta\,d\theta$

Limits: $x = 2,\ \theta = \frac{\pi}{2};\quad x = 0,\ \theta = 0.$

$$\therefore \int_{c_2}(P\,dx + Q\,dy) = \int_{\pi/2}^{0}\left\{4 - 4\sin\theta - \frac{4\sin^2\theta}{2\cos\theta}\right\}2\cos\theta\,d\theta$$

$$= 4\left[2\sin\theta - \sin^2\theta - \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_{\pi/2}^{0}$$

$$= 4\left[-\left(2 - 1 - \frac{\pi}{4}\right)\right] = \pi - 4$$

Finally

(3) c_3: $\quad x = 0;\quad dx = 0$

$$\therefore \int_{c_3}(P\,dx + Q\,dy) = \int_2^0 2y\,dy = \left[y^2\right]_2^0 = -4$$

$\therefore$ Collecting our three partial results

$$\oint_c(P\,dx + Q\,dy) = \frac{8}{3} + \pi - 4 - 4 = \pi - \frac{16}{3} \qquad (1)$$

That is one part done. Now we have to evaluate $\iint_S\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

$$P = x^2 + y^2 \qquad \therefore\ \frac{\partial P}{\partial y} = 2y$$

$$Q = x + 2y \qquad \therefore\ \frac{\partial Q}{\partial x} = 1$$

$$\therefore \iint_S\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \iint_S(1 - 2y)\,dy\,dx$$

It will be more convenient to work in polar coordinates, so we make the substitutions

$$x = r\cos\theta;\quad y = r\sin\theta;\quad dS = dx\,dy = r\,dr\,d\theta$$

$$\therefore \iint_S\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_0^{\pi/2}\int_0^2(1 - 2r\sin\theta)r\,dr\,d\theta$$

$$= \dots\dots\dots\dots$$

Complete it

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$$\boxed{\pi - \frac{16}{3}}$$

Here it is:

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_0^{\pi/2}\int_0^2 (r - 2r^2 \sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2} - \frac{2r^3}{3}\sin\theta\right]_0^2 \mathrm{d}\theta$$

$$= \int_0^{\pi/2}\left\{2 - \frac{16}{3}\sin\theta\right\}\mathrm{d}\theta$$

$$= \left[2\theta + \frac{16}{3}\cos\theta\right]_0^{\pi/2} = \pi - \frac{16}{3} \qquad (2)$$

So we have established once again that

$$\oint_c (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

And that brings us to the end of this particular Programme. We have covered a number of important sections, so check carefully down the **Revision summary** and the **Can You?** checklist, and then work through the **Test exercise** that follows. The **Further problems** provide valuable additional practice.

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Revision summary 18

1 *Line integrals*

(a) Scalar field V: $\int_c V\,\mathbf{dr}$

The curve c is expressed in parametric form.

$\mathbf{dr} = \mathbf{i}\,\mathrm{d}x + \mathbf{j}\,\mathrm{d}y + \mathbf{k}\,\mathrm{d}z$

(b) Vector field $\mathbf{F}$: $\int_c \mathbf{F}\cdot\mathbf{dr}$

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

$$\mathbf{dr} = \mathbf{i}\,\mathrm{d}x + \mathbf{j}\,\mathrm{d}y + \mathbf{k}\,\mathrm{d}z$$

$$\mathbf{F}\cdot\mathbf{dr} = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$$

2 *Volume integrals*

$\mathbf{F}$ is a vector field; V a closed region with boundary surface S.

$$\int_V \mathbf{F}\,\mathrm{d}V = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \mathbf{F}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

▶

3 *Surface integrals* (surface defined by $\phi(x, y, z) = \text{constant}$)

(a) Scalar field $V(x, y, z)$:

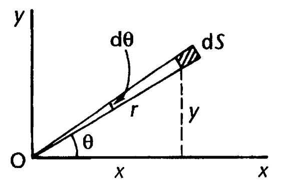
$$\int_S V\,d\mathbf{S} = \int_S V\hat{\mathbf{n}}\,dS; \qquad \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$$

(b) Vector field $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$:

$$\int_S \mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS; \qquad \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

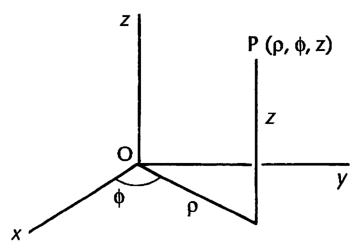
4 *Polar coordinates*

(a) Plane polar coordinates (r, θ)



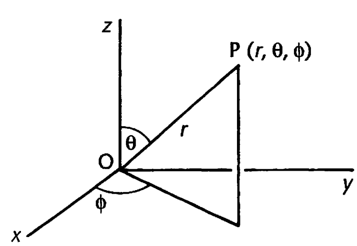
$$x = r\cos\theta; \quad y = r\sin\theta$$
$$dS = r\,dr\,d\theta$$

(b) Cylindrical polar coordinates (ρ, ϕ, z)



$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$
$$dS = \rho\,d\phi\,dz$$
$$dV = \rho\,d\rho\,d\phi\,dz$$

(c) Spherical polar coordinates (r, θ, ϕ)



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$dS = r^2\sin\theta\,d\theta\,d\phi$$
$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

5 *Conservative vector fields*

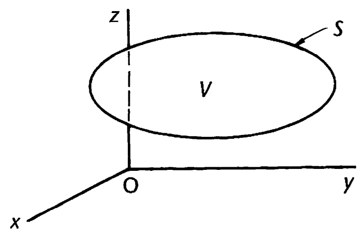
A vector field $\mathbf{F}$ is conservative if

(a) $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for all closed curves

(b) curl $\mathbf{F} = 0$

(c) $\mathbf{F} = \text{grad}\,V$ where V is a scalar.

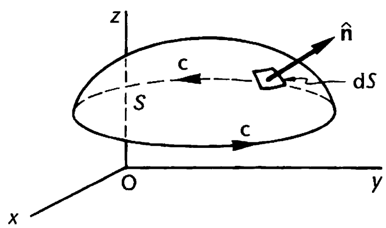
6 *Divergence theorem* (Gauss' theorem)



Closed surface S enclosing a region V in a vector field $\mathbf{F}$.

$$\int_V \operatorname{div} \mathbf{F}\,\mathrm{d}V = \int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$$

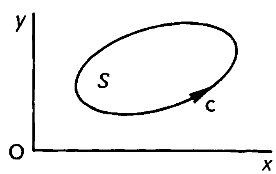
7 *Stokes' theorem*



An open surface S bounded by a simple closed curve c, then

$$\int_S \operatorname{curl} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \oint_c \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

8 *Green's theorem*



The curve c is a simple closed curve enclosing a plane space S in the x–y plane. P and Q are functions of both x and y.

Then $\displaystyle\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \oint_c (P\,\mathrm{d}x + Q\,\mathrm{d}y)$.

Can You?

85 Checklist 18

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: — **Frames**

- Evaluate the line integral of a scalar and a vector field in Cartesian coordinates? — 1 to 20

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate the volume integral of a vector field? — 21 to 27

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate the surface integral of a scalar and a vector field? 28 to 42

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine whether or not a vector field is a conservative vector field? 43 to 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply Gauss' divergence theorem? 52 to 63

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply Stokes' theorem? 64 to 68

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine the direction of unit normal vectors to a surface? 69 to 78

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply Green's theorem in the plane? 79 to 83

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 18

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1 If $V = x^3y + 2xy^2 + yz$, evaluate $\int_c V\,d\mathbf{r}$ between A (0, 0, 0) and B (2, 1, -3) along the curve with parametric equations $x = 2t$, $y = t^2$, $z = -3t^3$.

2 If $\mathbf{F} = x^2y^3\,\mathbf{i} + yz^2\,\mathbf{j} + zx^2\,\mathbf{k}$, evaluate $\int_c \mathbf{F}\cdot d\mathbf{r}$ along the curve $x = 3u^2$, $y = u$, $z = 2u^3$ between A (3, -1, -2) and B (3, 1, 2).

3 Evaluate $\int_V \mathbf{F}\,dV$ where $\mathbf{F} = 3\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$ and V is the region bounded by the planes $z = 0$, $z = 3$ and the surface $x^2 + y^2 = 4$.

4 If V is the scalar field $V = xyz^2$, evaluate $\int_S V\,d\mathbf{S}$ over the surface S defined by $x^2 + y^2 = 9$ between $z = 0$ and $z = 2$ in the first octant.

5 Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ over the surface S defined by $x^2 + y^2 + z^2 = 4$ for $z \geq 0$ and bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant where $\mathbf{F} = x\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$.

▶

6 Determine which of the following vector fields are conservative.

(a) $\mathbf{F} = (2xy + z)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (x + y^2)\mathbf{k}$

(b) $\mathbf{F} = (yz + 2y)\mathbf{i} + (xz + 2x)\mathbf{j} + (xy + 3)\mathbf{k}$

(c) $\mathbf{F} = (yz^2 + 3)\mathbf{i} + (xz^2 + 2)\mathbf{j} + (2xyz + 4)\mathbf{k}$.

7 By the use of the divergence theorem, determine $\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = x\mathbf{i} + xy\mathbf{j} + 2\mathbf{k}$, taken over the region bounded by the planes $z = 0$, $z = 4$, $x = 0$, $y = 0$ and the surface $x^2 + y^2 = 9$ in the first octant.

8 A surface consists of parts of the planes $x = 0$, $x = 2$, $y = 0$, $y = 2$ and $z = 3 - y$ in the region $z \geq 0$. Apply Stokes' theorem to evaluate $\int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$ over the surface where $\mathbf{F} = 2x\mathbf{i} + xz\mathbf{j} + yz\mathbf{k}$ where S lies in the $z = 0$ plane.

9 Verify Green's theorem in the plane for the integral

$$\oint_c \{(xy^2 - 2x)dx + (x + 2xy^2)dy\}$$

where c is the square with vertices at (1, 1), (−1, 1), (−1, −1) and (1, −1).

Further problems 18

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1 If $V = x^2yz$, evaluate $\int_c V\,d\mathbf{r}$ between A (0, 0, 0) and B (6, 2, 4)

(a) along the straight lines c_1: (0, 0, 0) to (6, 0, 0)
c_2: (6, 0, 0) to (6, 2, 0)
c_3: (6, 2, 0) to (6, 2, 4)

(b) along the path c_4 having parametric equations $x = 3t$, $y = t$, $z = 2t$.

2 If $V = xy^2 + yz$, evaluate to one decimal place $\int_c V\,d\mathbf{r}$ along the curve c having parametric equations $x = 2t^2$, $y = 4t$, $z = 3t + 5$ between A (0, 0, 5) and B (8, 8, 11).

3 Evaluate to one decimal place the integral $\int_c (xyz + 4x^2y)\,d\mathbf{r}$ along the curve c with parametric equations $x = 2u$, $y = u^2$, $z = 3u^3$ between A (2, 1, 3) and B (4, 4, 24).

4 If $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + 3xyz\mathbf{k}$, evaluate $\int_c \mathbf{F} \cdot d\mathbf{r}$ between A (0,2,0) and B (3,6,1) where c has the parametric equations $x = 3u$, $y = 4u + 2$, $z = u^2$.

5 $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + yz\mathbf{k}$. Evaluate $\int_c \mathbf{F}\cdot d\mathbf{r}$ between A (2, 1, 2) and B (4, 4, 5) where c is the path with parametric equations $x = 2u$, $y = u^2$, $z = 3u - 1$.

6 A unit particle is moved in an anticlockwise manner round a circle with centre (0, 0, 4) and radius 2 in the plane $z = 4$ in a force field defined as $\mathbf{F} = (xy + z)\mathbf{i} + (2x + y)\mathbf{j} + (x + y + z)\mathbf{k}$. Find the work done.

7 Evaluate $\int_V \mathbf{F}\,dV$ where $\mathbf{F} = \mathbf{i} - y\mathbf{j} + \mathbf{k}$ and V is the region bounded by the plane $z = 0$ and the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \geq 0$.

8 V is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and the surfaces $y = 4 - x^2$ $(z \geq 0)$ and $y = 4 - z^2$ $(y \geq 0)$.

If $\mathbf{F} = 2\mathbf{i} + y^2\mathbf{j} - \mathbf{k}$, evaluate $\int_V \mathbf{F}\,dV$ throughout the region.

9 If $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 2x\mathbf{k}$, evaluate $\int_V \mathbf{F}\,dV$ where V is the region bounded by the planes $y = 0$, $z = 0$, $z = 4 - y$ $(z \geq 0)$ and the surface $x^2 + y^2 = 16$.

10 A scalar field $V = x + y$ exists over a surface S defined by

$x^2 + y^2 + z^2 = 9$,

bounded by the planes $x = 0$, $y = 0$, $z = 0$ in the first octant. Evaluate $\int_S V\,d\mathbf{S}$ over the curved surface.

11 A surface S is defined by $y^2 + z = 4$ and is bounded by the planes $x = 0$, $x = 3$, $y = 0$, $z = 0$ in the first octant. Evaluate $\int_S V\,d\mathbf{S}$ over this curved surface where V denotes the scalar field $V = x^2yz$.

12 Evaluate $\int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ over the surface S defined by $2x + 2y + z = 2$ and bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant and where $\mathbf{F} = y^2\mathbf{i} + 2yz\mathbf{j} + xy\mathbf{k}$.

13 Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ over the hemisphere defined by $x^2 + y^2 + z^2 = 25$ with $z \geq 0$, where $\mathbf{F} = (x + y)\mathbf{i} - 2z\mathbf{j} + y\mathbf{k}$.

14 A vector field $\mathbf{F} = 2x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ exists over a surface S defined by $x^2 + y^2 + z^2 = 16$, bounded by the planes $z = 0$, $z = 3$, $x = 0$, $y = 0$. Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ over the stated curved surface.

▶

15 Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}$ is the vector field $x^2\mathbf{i} + 2z\mathbf{j} - y\mathbf{k}$, over the curved surface S defined by $x^2 + y^2 = 25$ and bounded by $z = 0$, $z = 6$, $y \geq 3$.

16 A region V is defined by the quartersphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, $y \geq 0$ and the planes $z = 0$, $y = 0$. A vector field $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j} + \mathbf{k}$ exists throughout and on the boundary of the region. Verify the Gauss divergence theorem for the region stated.

17 A surface consists of parts of the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 1$ in the first octant. If $\mathbf{F} = y\mathbf{i} + x^2z\mathbf{j} + xy\mathbf{k}$, verify Stokes' theorem.

18 S is the surface $z = x^2 + y^2$ bounded by the planes $z = 0$ and $z = 4$. Verify Stokes' theorem for a vector field $\mathbf{F} = xy\mathbf{i} + x^3\mathbf{j} + xz\mathbf{k}$.

19 A vector field $\mathbf{F} = xy\mathbf{i} + z^2\mathbf{j} + xyz\mathbf{k}$ exists over the surfaces

$$x^2 + y^2 + z^2 = a^2,\ x = 0 \text{ and } y = 0$$

in the first octant. Verify Stokes' theorem that $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$.

20 A surface is defined by $z^2 = 4(x^2 + y^2)$ where $0 \leq z \leq 6$. If a vector field $\mathbf{F} = z\mathbf{i} + xy^2\mathbf{j} + x^2z\mathbf{k}$ exists over the surface and on the boundary circle c, show that $\oint_c \mathbf{F}\cdot d\mathbf{r} = \int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$.

21 Verify Green's theorem in the plane for the integral

$$\oint_c \{(x - y)\,dx - (y^2 + xy)\,dy\}$$

where c is the circle with unit radius, centred on the origin.